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THE INITIAL DEVELOPMENT OF THE WKB SOLUTIONS OF LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS AND THEIR USE IN THE CONNECTION PROBLEM

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Summaries

In this paper we trace the early development of a method for finding the approximate solutions --called the WKB solutions--for a class of ordinary differential equations of second order. We also analyze the attempts made by the various contributors to this method to substantiate their results. These approximating solutions were subsequently shown to be asymptotic in the sense of Poincaré. Also presented and examined are the several methods used to deal with the "connection problem" which arises in the use of the WKB solutions.

Wir betrachten in dieser Arbeit die Entwicklung einer Methode um die annähernde Lösungen--die WKB Lösungen genannt--für eine Klasse gewöhlichen Differentialgleichungen zweiter Ordnung festzusetzen. Wir analysieren auch die Versuchungen vershiedener Beiträger zu dieser Methode die ihre Resultate mathematisch begründen wollten. Diese WKB Lösungen würden später bewiesen als asymptotische Lösungen im Sinne Poincarê's. Auch presentiert und untersucht sind die verschiedenen Methoden, das "connection problem" zu behandlen, welches alls der Benützung dieser WKB Lösungen ensteht.

INTRODUCTION

By the 1920's the theory of asymptotic series solutions was fairly complete for a large class of linear ordinary differential equations, but required the coefficients to be analytic functions. (A survey of these results is given in Schlissel [1977].) At this time the need arose to obtain an approximate solution for differential equations when the coefficients are not necessarily analytic. A method which achieved some fame was developed that yielded a one term approximation to the solution of a class of differential equations of type $w'' + \psi w = 0$ where ψ is a function of x and possibly also of a large parameter λ. The form of these approximating solutions causes them to be singular at certain points, although the actual solutions remain finite. The presence of these points, called turning points, gives rise to the connection problem--the determination of the approximating solution on one side of the turning point from that of the other side so that both represent the same actual solution. This paper traces the early development of this method and gives a critical analysis of the various contributors as to their motivations and how they satisfied the mathematical requirements of rigor.

The various researchers, whose works are discussed in this paper, attempted to justify their procedures, but these were at best token gestures. The contributors were in many instances physicists searching to obtain approximate solutions whose nature they had predetermined.

The approximating solutions discussed in this paper are of the same form, and were discussed and rediscovered by various researchers: George Green (1793-1841), Lord Rayleigh (1842-1909), Richard Gans (1880-1954), Harold Jeffreys (1891-), Gregor Wentzel (1898-), Hendrich A. Kramers (1884-1952) and Leon Brillouin (1889-1969).

After the results of Wentzel, Kramer and Brillouin were published, the approximating solutions were referred to as the WKB solutions, and after the belated discovery of Jeffrey's contribution, they were referred to as the JWKB solutions. To give Green and Gans proper credit, they should be referred to as solutions of type G. But in the interest of consistency with the major portion of the literature, the approximating solutions will be referred to as "WKB" solutions, even those that were found before that term was introduced.

The paper is divided into two sections. The first deals with the several approaches used to obtain the WKB solutions and the second with the various ways they were used to solve the connection problem. The notations of the various investigators discussed has been changed to make them uniform with each other. The function $\tilde{w}$ will denote a WKB solution of an equation satisfied by the function w. The bibliography consists of those items referred to in the text.

1. THE WKB SOLUTIONS

The equation for which "approximate solutions" are to be found, are of the form

$$\frac{d^2w}{dx^2} + \psi w = 0 \tag{1}$$

where ψ is either a function of x, $\psi = \psi(x)$, or a function of x

and a large positive parameter λ, $\psi = \psi(x,\lambda)$. The variable x is restricted to the finite interval $\Omega \equiv [\alpha,\beta]$ on the real x-axis and ψ is required to be a slowly varying function and have continuous second derivatives with respect to x on Ω.

The several special forms of equation (1) studied by the various investigators describe wave propagation. The approximation method discussed in this paper received prominence in the 1920's when attention focused on quantum mechanics. One of the important equations in this area of physics is the Schrödinger equation

$$\frac{d^2w}{dx^2} + \lambda^2\phi(x)w = 0 \tag{2}$$

with $\phi(x) \equiv 2m[E-V(x)]$, $\lambda^2 = 2\pi/h$,

where m is the mass of a particle, E is the energy level, $V(x)$ the potential field, λ is a large parameter and h is Planck's constant. The last equation also arises in the study of the harmonic oscillator.

When ψ is a function of x and λ, $\psi = \psi(x,\lambda)$, then these "approximate solutions" were used to solve the following eigenvalue problem: find solutions of the differential equation (1) and compatible λ values on the subinterval $[x_1,x_2] \subset \Omega$ satisfying the boundary conditions

$$w(x_1) = w(x_2) = 0, \tag{3}$$

where in addition,

$$\psi(x_1,\lambda) = \psi(x_2,\lambda) = 0,$$

$\psi \geq 0$ inside $[x_1,x_2]$, and $\psi < 0$ outside $[x_1,x_2]$. The last condition on ψ introduced an interesting difficulty in the use of the approximate solutions--the "connection problem"--and will be discussed in Section 2.

The general idea used to obtain the WKB solution can be traced back to Green [1837]. Green was studying wave propagation in an infinitely long and very narrow canal. The equation he had to solve was not of the form (1) but was the partial differential equation

$$\frac{\partial^2 w}{\partial x^2} + \left\{\frac{1}{\beta}\frac{\partial\beta}{\partial x} + \frac{1}{\gamma}\frac{\partial\gamma}{\partial x}\right\}\frac{\partial w}{\partial x} - \frac{1}{g\gamma}\frac{\partial^2 w}{\partial t^2} = 0 \tag{4}$$

where $\pm\beta$ and γ are slowly varying functions of x representing the sides and bottom of the canal respectively and g is the gravitational constant. Green was unable to find the exact solutions of this equation. But he observed that when β and γ are constants the differential equation reduces to the wave equation

(5) $$\frac{\partial^2 w}{\partial x^2} - \frac{1}{g\gamma}\frac{\partial^2 w}{\partial t^2} = 0$$

whose solution was given by d'Alembert as

$$w = H(t + x/\sqrt{g\gamma}) + h(t - x/\sqrt{g\gamma})$$

with the functions H and h determined by the initial conditions. He therefore assumed a solution of (4) in the form

(6) $$w = \eta(x)\ f[t + h(x)]$$

with $\eta(x)$ a slowly varying function of x. To determine the various functions in (6) that expression is substituted into (4). In the resulting expression all terms multiplied by various derivatives of f (which are assumed negligible) are dropped to yield the approximating solution

(7) $$w = \beta(x)^{-1/2}\ \gamma(x)^{-1/4}[f(t + \int^x \frac{d\xi}{\sqrt{g\gamma(\xi)}}) + F(t - \int^x \frac{d\xi}{\sqrt{g\gamma(\xi)}})].$$

Green considered (7) an approximate solution of (4) because it "almost" satisfies the differential equation, but did not give a precise meaning to "almost". This is obviously not a rigorous argument and is in general not true. Liouville [1837] studied the second order ordinary differential equation $(pw')' + (p_1\lambda - p_2)w = 0$ where p, p_1 and p_2 are functions of x, and λ is a large positive parameter. He found approximate solutions for this last equation similar in form to (7) but derived by a method that differed from Green's.

Rayleigh [1912] studied the equation

(8) $$\frac{d^2 w}{dx^2} + \phi(x)w = 0.$$

Prompted by the form of the solution if $\phi(x)$ were a constant, he made the assumption that

(9) $$w = \eta(x)\ \exp\left\{\pm i \int^x a(\zeta)\ d\zeta\right\}$$

(ζ a dummy variable) is a solution of the above equation. (Rayleigh used only the minus sign.) The functions $\eta(x)$ and $a(x)$ are determined by substituting (9) into (8) to obtain

(10) $$\frac{d^2\eta}{dx^2} + (\phi - a^2)\eta - 2i\ a^{1/2}\frac{d}{dx}(a^{1/2}\ \eta) = 0.$$

The amplitude function $\eta(x)$ is assumed to vary "slowly" with respect to x, so that the term η'' is of a higher order compared to η and η' and may be dropped in equation (10). Each of the remaining terms in (10) is then set equal to zero to yield

(11) $a(x) = \pm\, [\phi(x)]^{1/2}, \quad [a(x)]^{1/2}\eta(x) = C \quad (C = \text{const.}).$

These values substituted into (9) give the WKB solutions of (8)

$$\tilde{w}(x) = [\phi(x)]^{-1/4} \exp\left\{\pm\, i \int^{x} \sqrt{\phi(\zeta)}\,d\zeta\right\}. \tag{12}$$

Rayleigh recognized that the $\tilde{w}$ are oscillatory if $\phi(x) > 0$ and exponential if $\phi(x) < 0$. The only attempt by Rayleigh to verify his assumption concerning the form of the solution was to show that for a special case $w''+x^{-4}w = 0$ it coincided with the exact solution.

Several years later, in a paper that remained long unknown, Gans [1915] studied light propagation in inhomogeneous media governed by the equation (8) and found approximate solutions for it. He, like Rayleigh, used the case of constant $\phi(x)$ as a cue. He assumed that the WKB solutions for (8) when $\phi(x) > 0$ are

$$\begin{aligned} \tilde{w}_1 &= [\phi(x)]^{-1/4} \exp\left\{-i \int^{x} \sqrt{\phi(\zeta)}\,d\zeta\right\}, \\ \tilde{w}_2 &= [\phi(x)]^{-1/4} \exp\left\{+i \int^{x} \sqrt{\phi(\zeta)}\,d\zeta\right\}, \end{aligned} \tag{13}$$

and when $\phi(x) < 0$ they are

$$\begin{aligned} \tilde{w}_3 &= [-\phi(x)]^{-1/4} \exp\left\{-\int^{x} \sqrt{-\phi(\zeta)}\,d\zeta\right\}, \\ \tilde{w}_4 &= [-\phi(x)]^{-1/4} \exp\left\{+\int^{x} \sqrt{-\phi(\zeta)}\,d\zeta\right\}. \end{aligned} \tag{14}$$

The solutions $\tilde{w}_1$ and $\tilde{w}_2$ are of oscillatory character, while the solutions $\tilde{w}_3$ and $\tilde{w}_4$ are of exponential character. Another set of oscillatory WKB solutions can be obtained by the addition and subtraction of the two solutions in (13),

$$\begin{aligned} \tilde{w}_5 &= [\phi(x)]^{-1/4} \sin\left\{\int^{x} \sqrt{\phi(\zeta)}\,d\zeta\right\}, \\ \tilde{w}_6 &= [\phi(x)]^{-1/4} \cos\left\{\int^{x} \sqrt{\phi(\zeta)}\,d\zeta\right\}. \end{aligned} \tag{15}$$

Gans then proceeded to show why the WKB solutions $\tilde{w}_j$ ($j = 1,2,\ldots,6$) should be considered an approximate solution of equation (8). He substituted $\tilde{w}_j$ into equation (8) to obtain

$$\frac{d^2\tilde{w}}{dx^2} + \phi(x)\tilde{w} = \left[\frac{d^2}{dx^2}\left(\frac{1}{[\phi(x)]^{1/4}}\right)\right] \exp \pm\left\{i\int^{x} [\phi(\zeta)]^{1/2}\,d\zeta\right\}. \tag{16}$$

If $\phi(x)$ is such that $[d^2[\phi(x)]^{-1/4}/dx^2] \ll 1$, then the right side of (16) is negligible compared with the left side, and the $\tilde{w}_j$ "almost" satisfy equation (8). This, argued Gans, showed that the $\tilde{w}_j$ approximate the exact solutions. This argument is in general not true, as can be seen by examining the solutions of $w'' - 3w' + 2w = 0$. They "almost satisfy" the equation $w'' - 3w' + 2w = x^{-1}$, but they do not approach its solutions as $x \to \infty$.

The English mathematician Jeffreys [1925a,b,c] in his prize winning doctoral dissertation, dealt with the wave motion in an elliptical lake. He was interested in finding approximate solutions for equation (1) in the special form

$$\frac{d^2w}{dx^2} + [\lambda^2\phi(x) + \lambda\, \phi_1(x) + \phi_2(x)]w = 0 \tag{17}$$

for large values of the parameter λ. Jeffreys, unaware of Gans' work, followed the method of Horn [1899a,b] for equations with analytic coefficients, and postulated a formal solution

$$w = g(x)\, e^{\lambda h(x)} \left[1 + \frac{f_1(x)}{\lambda} + \frac{f_2(x)}{\lambda^2} + \dots\right] \tag{18}$$

for it, with $g(x)$, $h(x)$ and $f_i(x)$ functions of x only. Recursion formulas for these functions were obtained by formal substitution of the series into equation (17). The first three are

$$[h'(x)]^2 = \phi(x), \quad \frac{g'(x)}{g(x)} = \frac{\phi_1(x) - h''(x)}{2h'(x)}, \quad 2h'f' = \phi_2(x) - \frac{g'(x)}{g(x)}.$$

Jeffreys, assuming that the first term of the series (18) is a good approximation to w for large λ, dropped the remaining terms and obtained

$$\tilde{w} = [\phi(x)]^{-1/4} \exp\left\{\pm \int_0^x \left[\lambda\sqrt{\phi(\zeta)} - \frac{\phi_1(\zeta)}{2\sqrt{\phi(\zeta)}}\right] d\zeta\right\}. \tag{19}$$

The lower limit of integration in (19) is arbitrary and serves as an additive constant. The term $\phi_2(x)$ has no effect in this order of approximation. When $\phi_1 = \phi_2 = 0$, these $\tilde{w}$'s, for $\phi(x)>0$ reduce to

$$\begin{aligned} \tilde{w}_7 &= [\phi(x)]^{-1/4} \exp\left\{-i\lambda\int^x\sqrt{\phi(\zeta)}\,d\zeta\right\} \\ \tilde{w}_8 &= [\phi(x)]^{-1/4} \exp\left\{i\lambda\int^x\sqrt{\phi(\zeta)}\,d\zeta\right\} \end{aligned} \tag{20}$$

and for $\phi(x) < 0$ they reduce to

$$\begin{aligned} \tilde{w}_9 &= [-\phi(x)]^{-1/4} \exp\left\{-\lambda\int^x\sqrt{-\phi(\zeta)}\,d\zeta\right\} \\ \tilde{w}_{10} &= [-\phi(x)]^{-1/4} \exp\left\{\lambda\int^x\sqrt{-\phi(\zeta)}\,d\zeta\right\}. \end{aligned} \tag{21}$$

The solutions $\tilde{w}_9$ and $\tilde{w}_{10}$ are of exponential character while the solutions $\tilde{w}_7$ and $\tilde{w}_8$ are oscillatory character. Another set of oscillatory WKB solutions can be obtained by the addition and subtraction of the solutions in (20),

$$\tag{22} \begin{aligned} \tilde{w}_{11} &= [\phi(x)]^{-1/4} \sin[\lambda \int^x \sqrt{\phi(\zeta)}d\zeta], \\ \tilde{w}_{12} &= [\phi(x)]^{-1/4} \cos[\lambda \int^x \sqrt{\phi(\zeta)}d\zeta]. \end{aligned}$$

Jeffreys claimed that the error incurred by using any of the approximate solutions $\tilde{w}$ is $O(\lambda^{-1})$, but gave no convincing proof. The solutions obtained by Jeffreys should be compared to those obtained by Gans. Jeffreys in [1915] and Fowler [1920] obtained a similar solution but by a less explicit method.

In the late 1920's, almost simultaneously but independent of each other, the three physicists Wentzel, Kramer and Brilliouin found approximate solutions for the Schrödinger equation (2). Wentzel [1926], for $\phi(x) > 0$ used the substitution $w = \exp\{2\pi i\lambda \int^x u(\zeta)\} d\zeta$ to reduce equation (2) to the Riccati equation

$$\tag{23} \frac{2\pi i}{\lambda}\frac{du}{dx} = \phi - u^2.$$

Prompted by the power series method for finding the actual solutions of differential equations Wentzel assumed a solution of (23) in the form

$$\tag{24} u = \sum_{\nu=0}^{\infty} \left(\frac{1}{\pi\lambda}\right)^{\nu} u_\nu(x).$$

For large λ the left side of (23) is small compared to the right side, so that $u_0 = \pm\phi_0^{1/2}$. This series substituted into the Riccati equation, yielded recursion formulas for the u_ν,

$$u'_{\nu-1} + \sum_{\alpha=0}^{\nu} u_\alpha u_{\nu-\alpha} = 0,$$

that is

$$u_0 = \pm\phi_0^{1/2}, \quad u_1 = \frac{-u_0'}{2u_0}, \quad u_2 = \frac{-u_1' + u_1^2}{2u_0}, \ \ldots .$$

The $u_\nu(x)$ can be found as long as $u_0(x) \neq 0$. Wentzel recognized that the formal series (24) does not necessarily converge even for large values of λ but assumed that it was an asymptotic solution for the Ricatti equation. He retained only the first two terms of the formal series and obtained the WKB solutions

(25) $$\tilde{w} = [\phi(x)]^{-1/4} \exp\left\{\pm 2\pi i\ \lambda \int^{x} \sqrt{\phi(\zeta)}\ d\zeta\right\}$$

for equation (2). Within the span of a few months, Brillouin [1926a, b] used the same technique to obtain the WKB solutions and indicated its extension to several variables.

At this same time Kramer [1926] independently studied equation (2) for the boundary value problem discussed at the beginning of this section. To obtain a solution, Kramer proceeded in a manner different from that of Wentzel and Brillouin. He first assumed that the solution of equation (2) should have the form

(26) $$w = g(x)\ \cos[\lambda f(x)],$$

where $g(x)$ is a "smooth" function, with it and $f(x)$ to be determined. Since $w = 0$ at the end points of the interval, and since if ϕ is constant, the wave length would be $(2\pi/\lambda)\phi^{-1/2}$, it follows that as an approximation, $f(x + [2\pi/\lambda]\phi^{-1/2}) - f(x) = 2\pi$. Or, using the first two terms of the Taylor series expansion for f, $(2\pi/\lambda)\phi^{-1/2}\ f(x) = 2\pi$. To determine $g(x)$ he replaced $\phi(x)$ by the first two terms of its Taylor expansion $\phi(x) = \phi(x_0) + \phi'(x_0)(x - x_0)$ with x_0 some point in the interval $\Omega \equiv [\alpha,\beta]$ on the x-axis whereby the original equation took the form

(27) $$\frac{d^2w}{dx^2} + \lambda^2[\phi(x_0) + \phi'(x_0)(x-x_0)]w = 0.$$

The solution of the last equation can be written explicitly by the method of variation of parameters. Dropping higher order terms, and comparing the resultant expression with (26) indicated that $g = \phi^{-1/4}$. Consequently when λ is large, the approximate solution for (2) is

(28) $$\tilde{w} = [\phi(x)]^{-1/4}\ \cos\ [\lambda \int_{x_0}^{x} \sqrt{\phi(\zeta)}\ d\zeta].$$

Kramer's method for obtaining the approximate solution depended on the possibility of replacing $\phi(x)$ in the differential equation by a linear expression in an interval of the size of the wave length and is certainly not possible for a wide class of $\phi(x)$. From its derivation it is seen that $\tilde{w}$ given by (28) "approximately" satisfies the original differential equation (2).

The WKB solutions obtained by the various investigators consist of one term and have the same form. And it was assumed by them that an appropriate combination of the WKB solutions would approximate any particular solution of (1). To indicate that the WKB solutions approximate the actual solutions it was shown that they almost satisfy the differential equation. This method was intuitive but not rigorous. What remained unresolved and unproven at this time was the basic question: "To what degree do the WKB solutions approximate the actual one's?"

With time the WKB solutions became a topic of interest in its own right and underwent closer analysis. The question posed in the last paragraph was answered later by Olver [1954a, b] who showed that the WKB solutions are the first term of the asymptotic series solutions of the differential equation. It was subsequently extended to several space variables [Heading 1965] and coefficients having certain singularities [Larkin and Sanchez 1971].

2. THE CONNECTION PROBLEM

There is a natural restriction in the use of the WKB solutions caused by their form which gave rise to a very interesting problem. In the neighborhood of a zero of $\phi(x)$ the factor $[\phi(x)]^{-1/4}$ causes the WKB solution to become singular while the actual solutions remain finite. Therefore the relationship between the WKB solution and the true solution breaks down in a neighborhood of such a point. This point is called a "transition" or "turning point" of order n if the function $\phi(x)$ has a zero of order n there. This point is called a "transition point", since for the cases initially discussed, the WKB solution changes from oscillatory to exponential behavior as it crosses the zero point. The transition point is also called a "turning point" because physically part of the energy represented by the solution is reflected at the zero of $\phi(x)$. Should the x-interval contain a zero of $\phi(x)$ then the problem of how the WKB solutions from one side of the turning point are connected to the WKB solution on the other side of the turning point must be solved. In particular, for equation (8) if the general WKB solution of the left of the turning point is $C_1\tilde{w}_1 + C_2\tilde{w}_2$ and to the right of the turning point it is $D_3\tilde{w}_3 + D_4\tilde{w}_4$, it becomes necessary to determine the relationships between the constants C_1, C_2, D_3 and D_4 in order that the WKB solutions approximate the same actual solution on both sides of the turning point. Likewise for equation (3) it would be be necessary to relate $C_1\tilde{w}_9 + C_2\tilde{w}_{10}$ to $D_1\tilde{w}_{11} + D_2\tilde{w}_{12}$ across the turning point. This is called the "connection" or "continuation" problem.

In several of the propagation problems referred to in section 1 turning points occurred in the interior of the interval or in the case of the eigenvalue problem at both end points of the interval. It was thus of practical interest to study this problem.

Various approaches were made to deal with the connection problem, some by people discussed in section 1. The first was by Rayleigh [1912]. In trying to understand how the WKB solution can be continued across a turning point taken to be at $x = 0$, he studied the special case when the original equation took the form of the Airy equation in the form $w'' + (\pi/2)^{2/3}\, x\, w = 0$. A particular solution of this equation, the Airy integral $w(x) = \int_0^\infty \cos[(\pi/2)(\xi^3 - x\xi)]d\xi$, was shown by Stokes [1856] to have its asymptotic solutions for large x oscillatory when $x > 0$ and exponential when $x > 0$.

Gans developed a method for dealing with the connection problem which unfortunately remained unknown for some time and was later rediscovered by Jeffreys. There are some variations between the two techniques, and they complement each other. Gans' method is presented first followed by Jeffreys' variant. Gans studied equation (8) with $\phi(x) > 0$ for $x < 0$ and $\phi(x) < 0$ for $x > 0$. The particular physical phenomena under examination was wave propagation in a medium for $x < 0$ which is reflected at $x = 0$. In terms of the connection problem, he examined how a general oscillatory WKB solution of (8), $C_1\tilde{w}_1 + C_2\tilde{w}_2$ for $x < 0$, is connected across the turning point $x = 0$ of order 1 to the exponentially small WKB solution $D_3\tilde{w}_3$ in $x > 0$. Because the $\tilde{w}_4$ solution is exponentially large for $x >> 0$, it represents a solution with unbounded energy, and is therefore discarded on physical grounds.

For the first step, Gans divided the interval $\Omega \equiv [\alpha,\beta]$ into three subintervals by the points $x = \pm\, a$ (Figure 1).

Figure 1

The intervals $[\alpha,-a]$, $[-a,+a]$ and $[+a,\beta]$ will be referred to as the intervals I, II, and III respectively. He assumed that the quantity a can be chosen to satisfy the following conditions:

(i) the quantity a is so small that in the interval II the function $\phi(x)$ can be "approximated" by the first non-zero term of its Taylor series,

$$\phi(x) \sim \phi'(0)x = -Kx$$

where $-K = \phi'(0)$ with $K > 0$ since $\phi'(0) > 0$. This can obviously be done if $[\phi(0)/2\phi_0(x)]$ is "small" in II.

(ii) the quantity a is so large that the WKB solutions $\tilde{w}_1,\tilde{w}_2$ and $\tilde{w}_3,\tilde{w}_4$ are valid in the intervals I and III respectively. Moreover at $x = -a$ the quantity $(2/3)\sqrt{-Kx^3}$ and at $x = +a$ the quantity $(2/3)\sqrt{Kx^3}$ are each large compared to unity.

In interval II, Gans used condition (i) to replace equation (8) by the Airy equation

$$\bar{w}'' - Kx\bar{w} = 0. \tag{29}$$

Its solution can be expressed in terms of Hankel functions,

$$\bar{w}_1 = \sqrt{-x}\, H^{(1)}_{1/3}\left\{(2/3)\sqrt{-Kx^3}\right\},\quad \bar{w}_2 = \sqrt{-x}\, H^{(2)}_{1/3}\left\{(2/3)\sqrt{-Kx^3}\right\}. \tag{30}$$

He then showed how the solutions $\tilde{w}_1$ and $\tilde{w}_2$ of interval I can be connected to the solutions $\tilde{w}_3$ and $\tilde{w}_4$ of interval III by the use of $\bar{w}_1$ and $\bar{w}_2$ of interval II. Gans first showed how $\tilde{w}_1$ and $\tilde{w}_2$ can be related to $\bar{w}_1$ and $\bar{w}_2$. The solutions $\bar{w}_1$ and $\bar{w}_2$ can be approximated at $x = -a$ by the use of the first term of the asymptotic expansion of the Hankel functions for large argument,

$$\begin{aligned} \bar{w}_1 &\sim \sqrt{3/\pi}\,[-Kx]^{-1/4} \exp\left\{i[(2/3)\sqrt{-Kx^3} - 5\pi/12]\right\} \\ \bar{w}_2 &\sim \sqrt{3/\pi}\,[-Kx]^{-1/4} \exp\left\{-i[(2/3)\sqrt{-Kx^3} - 5\pi/12]\right\}. \end{aligned} \tag{31}$$

In interval I, the WKB solutions $\tilde{w}_1$ and $\tilde{w}_2$ of the original equation (8) can be approximated at $x = -a$, because of condition (i), by

$$\begin{aligned} \tilde{w}_1 &\simeq [-Kx]^{-1/4} \exp\left\{i\ (2/3)\sqrt{-Kx^3}\right\}, \\ \tilde{w}_2 &\simeq [-Kx]^{-1/4} \exp\left\{-i\ (2/3)\sqrt{-Kx^3}\right\}. \end{aligned} \tag{32}$$

According to condition (i) the Airy equation (29) approximates the actual equation (8) at $x = -a$, and within this approximation, Gans assumed that their solutions should coincide at $x = -a$. An inspection of the forms of approximation (31) and (32) indicates that at $x = -a$, the solutions $\tilde{w}_1$ and $\tilde{w}_2$ can be expressed as a multiple of $\bar{w}_1$ and $\bar{w}_2$ respectively

$$\tilde{w}_1 = \sqrt{\pi/3}\,\exp\{5\pi i/12\}\bar{w}_1,\quad \tilde{w}_2 = \sqrt{\pi/3}\,\exp\{-5\pi i/12\}\bar{w}_2. \tag{33}$$

The same procedure is then applied to connect the exponentially small solution $\tilde{w}_3$ from interval III and the solutions $\bar{w}_1$ and $\bar{w}_2$ of the Airy equation from interval II at $x = +a$. Near

$x = +a$, by use of the asymptotic form of the Hankel function for large argument,

$$(34)\quad \bar{w}_1 = \sqrt{x}\, H_{1/3}^{(1)}\left[2i/3\sqrt{Kx^3}\right] \sim [Kx]^{-1/4}\sqrt{3/\pi}\,\exp\left\{-2\pi i/3\right\} \exp\left\{-(2/3)\sqrt{Kx^3}\right\}$$

and because of condition (i),

$$(35)\quad \tilde{w}_3 \approx [Kx]^{-1/4}\exp\left\{-2/3\sqrt{Kx^3}\right\}.$$

A comparison of the relations (34) and (35) yields

$$(36)\quad \tilde{w}_3 = \sqrt{\pi/3}\,\exp\left\{2\pi i/3\right\}\bar{w}_1.$$

Using the relations (33) and (36), Gans obtained the continuation of the general WKB solution from interval I into interval II as

$$(37)\quad C_1\tilde{w}_1 + C_2\tilde{w}_2 \rightarrow C_1\sqrt{\pi/3}\,\exp\left\{5\pi i/12\right\}\bar{w}_1 + C_2\sqrt{\pi/3}\,\exp\left\{-5\pi i/12\right\}\bar{w}_2.$$

In turn, the WKB solution $D\tilde{w}_3$ from interval III is continued into interval II as

$$(38)\quad D\sqrt{\pi/3}\,\exp\left\{2\pi i/3\right\}\bar{w}_1 \leftarrow D\tilde{w}_3.$$

The two continuations (37) and (38) into interval II, because they involve the exact solutions $\bar{w}_1$ and $\bar{w}_2$ of the Airy equation (29) and are to represent the same actual solution, must be equal at $x = 0$:

$$(39)\quad \begin{aligned} &C_1\sqrt{\pi/3}\,\exp\,[5\pi i/12]\bar{w}_1(0) + C_2\sqrt{\pi/3}\,\exp\,[-5\pi i/12]\bar{w}_2(0) \\ &\quad = D\,\exp\,[2\pi i/3]\bar{w}_1(0). \end{aligned}$$

Since the $\bar{w}_1$ and $\bar{w}_2$ can be expressed in terms of Hankel functions $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$, Gans used a result from Jahnke and Emde [1909, 90] to show that for small values of x,

$$\sqrt{-K^{1/3}x}\; H_{1/3}^{(1)}\left[2/3\sqrt{-Kx^3}\right] \approx \frac{2\,\exp\,\{-i\pi/2\}}{\sqrt{3}}\left[\exp\{-i\pi/3\}\varepsilon_1 K^{1/3}x + \varepsilon_2\right]$$

$$\sqrt{-K^{1/3}x}\; H_{1/3}^{(2)}\left[2/3\sqrt{-Kx^3}\right] \approx \frac{2\,\exp\,\{i\pi/2\}}{\sqrt{3}}\left[\exp\{i\pi/3\}\varepsilon_1 K^{1/3}x + \varepsilon_2\right]$$

with $\varepsilon_1 = 0.7762, \ldots, \varepsilon_2 = 1.065\ldots$. These approximations used in (39) yield

$$(40)\quad C_1 \exp\,[5\pi i/12]\,\exp\,[-i\pi/2]\,[\exp\{-i\pi/3\}\varepsilon_1 K^{1/3}x + \varepsilon_2]$$

$$+ C_2 \exp\,[-5\pi i/12]\,\exp\,[i\pi/2]\,[\exp\{i\pi/3\}\varepsilon_1 K^{1/3}x + \varepsilon_2]$$

$$= D \exp\,\{2\pi i/3\}\,\exp\,\{i\pi/3\}\,[\varepsilon_1 K^{1/3}x - \varepsilon_2],$$

which in turn give the compatible values of the constants,

$$(41)\qquad C_2 = C_1 \exp\,\{i\pi/2\},\quad D = C_1 \exp\{i\pi/4\}.$$

Using these constants in the relations (37) and (38), Gans obtained his final result:

$$(42)\qquad \tilde{w}_1 + \exp\,\{i\pi/2\}\,\tilde{w}_2 \rightarrow \exp\,\{i\pi/4\}\,\tilde{w}_3\,,$$

where the expression on the left is a solution in I and the expression on the right is its continuation across the turning point $x = 0$ into II.

Jeffreys' variation, discussed next, allowed the constants C_1 and C_2 to be chosen independently of each other. Jeffreys employed Gans' approach (with some differences) to deal with the connection problem for equation (2), $w'' + \lambda^2\phi(x)w = 0$, with the turning point of order 1 located at $x = 0$ where $\phi(x) < 0$ for $x < 0$, $\phi(x) > 0$ for $x > 0$ and $\phi'(0) > 0$. In particular he studied how $\tilde{w}_9$ and $\tilde{w}_{10}$ each are to be continued across the turning point in terms of $\tilde{w}_{11}$ and $\tilde{w}_{12}$ (see (21) and (22)).

Jeffreys divided the interval $[\alpha,\beta]$ into the neighborhood of $x = 0$ and its complement and implicitly used the conditions stated by Gans. For clarity we impose on Jeffreys the intervals and conditions explicitly stated by Gans. This in no way affects the description of Jeffreys' procedure.

In interval II Jeffreys replaced the original equation (2) by

$$(43)\qquad \frac{d^2w}{dx^2} + Kxw = 0,$$

with $K = \lambda^2\phi'(0) > 0$. This is a form of Airy's equation. Its general solution for $x < 0$ is

$$(44)\quad w = A\sqrt{x}\, I_{-1/3}\left((2/3)\sqrt{Kx^3}\right) + B\sqrt{x}\, I_{1/3}\left((2/3)\sqrt{Kx^3}\right)$$

which for $x > 0$ becomes

$$(45)\quad w = A\sqrt{x}\, J_{-1/3}\left((2/3)\sqrt{Kx^3}\right) - B\sqrt{x}\, J_{1/3}\left((2/3)\sqrt{Kx^3}\right).$$

(Lommel [1868, 112]). The $J_{\pm 1/3}$ and $I_{\pm 1/3}$ are Bessel functions of the first and second kind respectively. Because of condition (i), the WKB solution $\tilde{w}_9$ from interval I can be approximated at $x = -a$ by

$$\text{(46)}\qquad \begin{aligned}\tilde{w}_7 &= [-\phi(x)]^{-1/4} \exp\left\{-\int^x \sqrt{-\phi(\zeta)d\zeta}\right\}\\ &\simeq [-Kx]^{-1/4} \exp\left\{-(2/3)\sqrt{-Kx^3}\right\}.\end{aligned}$$

Jeffreys assumed that this approximation should coincide at $x = -a$ with the asymptotic form of the solutions of the Airy equation (43) from interval II. An examination of the asymptotic form of the Bessel functions $I_{\pm 1/3}$ at $x = -a$ [1] shows that the appropriate solution (44) to coincide with the right side of (46) is

$$\text{(47)}\qquad \begin{aligned}\tilde{w}_7 &\simeq [-Kx]^{-1/4} \exp\left\{-(2/3)\sqrt{Kx^3}\right\}\\ &\simeq (2\sqrt{x}/\sqrt{3})\left[I_{-1/3}\left((2/3)\sqrt{Kx^3}\right) - I_{1/3}\left((2/3)\sqrt{Kx^3}\right)\right]\end{aligned}$$

where $\alpha = (3/\pi)^{1/2}K^{-1/4}$. The expression on the right becomes for $x > 0$, by (45),

$$(2\sqrt{x}/a\sqrt{3})\left[J_{-1/3}(2/3)\sqrt{Kx^3} + J_{1/3}(2/3)\sqrt{Kx^3}\right].$$

The first term of the asymptotic expansion of the last expression at $x = +a$ is

$$2[\phi(x)]^{-1/4} \cos\left(\pi/4 - \int_0^x \sqrt{\phi(\zeta)}\, d\zeta\right).$$

This expression can by (22) be expressed as a linear combination of $\tilde{w}_{11}$ and $\tilde{w}_{12}$ and gives

$$\text{(48)}\qquad \tilde{w}_9 \leftrightarrow \sqrt{2}\,\tilde{w}_{11} + \sqrt{2}\,\tilde{w}_{12}$$

as the continuation of $\tilde{w}_9$ into the interval III.

The same procedure is used for the exponentially large solution $\tilde{w}_{10}$. In a neighborhood of $x = -a$,

$$\text{(49)}\qquad \tilde{w}_{10} = [-\phi(x)]^{-1/4}\exp\left\{\int_0^x \sqrt{-\phi(\zeta)d\zeta}\right\} \simeq (2/\alpha)x^{1/2}\, I_{-1/3}\left((2/3)\sqrt{Kx^3}\right).$$

This last expression for $x > 0$ becomes $(2/\alpha)\sqrt{x}\, J_{-1/3}\left((2/3)\sqrt{Kx^3}\right)$ which at $x = +a$ has the asymptotic expansion

$$\text{(50)}\qquad 2[\phi(x)]^{-1/4} \sin\left(5\pi/12 + \int_0^x \sqrt{\phi(\zeta)}\, d\zeta\right).$$

So that finally,

$$\text{(51)}\qquad \tilde{w}_{10} \leftrightarrow 2\tilde{w}_{11} \sin 5\pi/12 + 2\tilde{w}_{12} \cos 5\pi/12.$$

Relations (48) and (51) are the connection formulas obtained by Jeffreys, where the expressions to the left of the arrows are valid for $x < -a$, and the expressions to the right of the arrows are the corresponding expressions for $x > +a$.

The Gans-Jeffreys method was applied by Goldstein [1928] to equation (2) when $\phi(x)$ has a zero of arbitrary order at $x = 0$. When $\phi(x)$ has a zero of order $2n$ at $x = 0$ and is negative on both sides of $x = 0$, then in interval II the original equation is replaced by $w'' + K^{2n+2} x^{2n} w = 0$ where $K^{2n+2} = \lambda^2 (1/(2n)!) (d^{2n}\phi(0)/dx^{2n})$. The solutions of the last equation in interval II used for the continuation process are $(\pm x)^{1/2} J_{\pm 1/(2n+2)}(\theta)$ and $(\pm x)^{1/2} I_{\pm 1/(2n+n)}(\theta)$, with $\theta = (Kx)^{n+1}/(n+1)$. The method was further extended by Goldstein [1932] to the equation $w'' + [\lambda^2\phi(x) + \lambda\phi_1(x)]w = 0$ when $\phi(x)$ has a zero of order 2 at $x = 0$. In this case the differential equation reduces to Weber's equation (also known as the parobolic equation), $w'' + (n + 1/2 - x^2/4)w = 0$, in interval II. We refer the reader to the original papers for details.

We briefly recapitulate the Gans-Jeffreys procedure:

(1) In interval II the original equation is replaced by an equation solvable in terms of Bessel functions.

(2) The WKB solutions from interval I are approximated in the neighborhood of $x = -a$.

(3) These approximations are matched with the asymptotic expansions of the proper combination of Bessel functions from interval II.

(4) This proper combination of the Bessel functions is then continued to $x = +a$ where its asymptotic form is found, and then matched with the approximations of the WKB solutions from interval III.

There are two differences between the procedures of Gans and Jeffreys. The first is that Gans used Hankel functions while Jeffreys used the Bessel functions as solutions in interval II. The second is that Gans continued the solutions from intervals I and III into interval II simultaneously and then used a compatibility procedure at $x = 0$, while Jeffreys continued the WKB solutions from interval I into II and then into III omitting the compatibility procedure at $x = 0$. He used instead a continuation process for the Bessel functions at $x = 0$. Jeffreys' use of the Bessel function entails some question as to how they are continued from $x < 0$ to $x > 0$, since they are multi-valued. Gans did not encounter this difficulty but had to rely on the availability of the constants ε_1 and ε_2, used in (40). In addition Jeffreys continued both the large and small solutions, while Gans considered only the exponentially small solution.

Jeffreys' use of the double headed arrow in the connection formulas caused controversy with regard to its meaning and usage. The formula has been misinterpreted to mean that the connection formulas work from right to left as well as from left to right. But reflection on the procedure used shows that they connect from left to right only. But there is an inherent difficulty in Jeffreys' result when it is applied to a general solution. A general solution of equation (1) in the interval I can be an appropriate combination of the WKB solutions or only by a multiple of the exponentially large WKB solution. However, the continuation of these two representations into interval III will each be different, thereby introducing an ambiguity as to what the proper continuation should be. A convention to deal with this difficulty was introduced by Goldstein [1928].

The Gans-Jeffreys method is based on a clever and intuitive application of the idea of approximation and it is precisely this which forms the basis for its critique. Assuming the method has validity, there are questions that need to be resolved:

(i) can the interval Ω be divided into three required regions?

(ii) can the original equation in region II be replaced by a simpler equation?

The procedure piles approximation upon approximation without stating the error committed at any step and without giving cumulative error involved which makes the technique from a rigorous point of view worthless.

Another approach to the connection problem was made by Kramer [1926] when he studied the equation (2). He employed a method similar to Rayleigh's. We have already seen in section 1 that to obtain the WKB solutions Kramer "reduced" the original equation (2) to the form $w'' + \lambda^2[\phi(x_0) + \phi'(x_0)(x-x_0)]w = 0$. For the boundary value problem mentioned previously (see (3)), the left end point $x = x_1$ of the interval is a zero of $\phi(x)$ at which this last equation becomes

$$w'' + \lambda^2 K\xi w = 0, \quad \phi'(x_1) = K, \quad x-x_1 = \xi, \tag{52}$$

a form of Airy's equation. Solutions for this equation can be written as

$$w = C \int_{\Gamma_i} \exp\left\{(\lambda^2 K)^{1/3}\, \xi t + 1/3\, t^2\right\} dt \tag{53}$$

with Γ_i being any of three paths depicted in Figure 2.

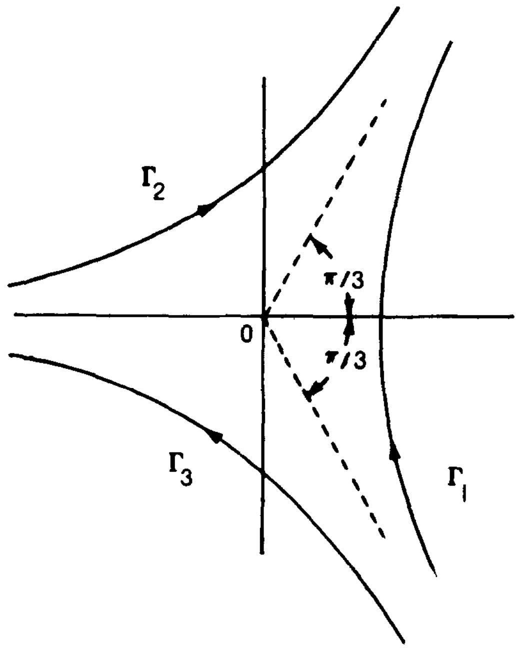

Figure 2

The physical phenomenon of wave propagation examined by Kramers, required that the solution be bounded for all x. The Debye saddle point method [Jeffreys and Jeffreys 1946, 504] applied to the integral in (53) gives the asymptotic behavior of w for each of the three paths of integration and shows that w has the required behavior on Γ_1 which is the path taken. For this path, the first term of the asymptotic expansion of w for large $(K\lambda^2)^{1/3}|\xi|$ is

(54) $$w \sim \sqrt{\pi}\left[(\lambda^2 K)^{1/3}|\xi|\right]^{-1/4} \exp\left\{-(2/3)(\lambda^2 K)^{1/2}|\xi|^{3/2}\right\}, \quad \xi < 0,$$

(55) $$w \sim 2\sqrt{\pi}\,[(\lambda^2 K)^{1/3}\xi]^{-1/4} \cos\left((2/3)(\lambda^2 K)^{1/2}\xi^{3/2} - \pi/4\right), \quad \xi > 0.$$

For x in a small neighborhood of x_1 $(\xi \simeq 0)$ expression (55) is compared to (28) to yield

$$w = [\lambda^2 K]^{-1/4}\,\xi^{-1/4} \cos\,(2/3)(\lambda^2 K)^{1/2}\xi^{3/2} - \beta\ ,$$

where β is an integration constant to be determined. From this follows that the continuation of the exponentially small solution from $x < x_1$ to the appropriate oscillatory solution for $x > x_1$ is $\tilde{w} = [\phi(x)]^{-1/4} \cos\,[\lambda \int_{x_1}^{x}\sqrt{\phi(\zeta)}\,d\zeta - \pi/4]$. Kramer then continued this solution to the other boundary point x_2 and across it. Kramer's method required $\phi(x)$ to be "almost linear" in the interval considered, but even this loosely given condition cannot always be applied. His use of the Airy integral avoided the matching process encountered in the Gnas-Jeffreys method but made his approach inapplicable to higher order turning points. This technique because of its limited applicability and lack of error estimates is of minimal value.

Another interesting method for treating the connection problem was developed by Zwaan [1929], a student of Kramer's. He studied a propagation problem governed by equation (2) where $\phi(x)$ did not satisfy the requirement of Kramer's method for crossing a turning point. He had to connect an exponential WKB solution across a turning point of order one located at $x = 0$ to an oscillatory WKB solution. Specifically, he wished to connect the WKB solution given for $x < 0$,

$$\tilde{w}_\ell = C_1[-\phi(x)]^{-1/4} \exp\left\{\lambda \int_0^x \sqrt{-\phi(\zeta)}d\zeta\right\} + C_2[-\phi(x)]^{-1/4} \exp\left\{-\lambda \int_0^x \sqrt{-\phi(\zeta)}d\zeta\right\} \tag{56}$$

across the turning point $x = 0$, to the solution given for $x > 0$,

$$\tilde{w}_r = B[\phi(x)]^{-1/4} \cos\left[\lambda \int_0^x \sqrt{\phi(\zeta)}d\zeta + \beta\right] \tag{57}$$

with C_1, C_2, B and β real constants. Since on physical grounds the solution must be bounded, we must have $C_1 = 0$ for $x < 0$.

In order to connect the WKB solutions, Zwaan utilized the process of analytic continuation, in an interesting manner [Ahlfors 1953, 210]. He continued equation (2) and its WKB solutions into the complex plane passing around the turning point. To do so, Zwaan, assumed:

(i) $\phi(x)$ can be "approximated" (in some sense left undefined) by an analytic function in a disc of radius R about $x = 0$;

(ii) the radius R should be large enough so that the WKB solutions are useable at $x = \pm R$.

On the basis of assumption (i) Zwaan continued $\phi(x)$ and with it equation (2) from $x = -R$ around the upper arc L of the circle about $x = 0$ to the real axis $x = +R$ (Figure 3).

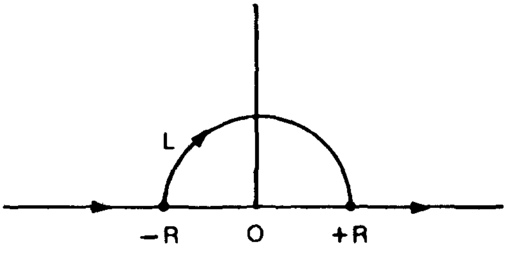

Figure 3

Using assumption (ii), he continued the WKB solution (56) from $-R$ along L to $+R$ and assumed that it retained its property as a WKB solution along L. Since the continuation of the WKB solution (56), at $x = +R$ must be oscillatory and real, the constant C_1 must undergo a jump from zero to some constant at some point along the arc L, which, to make the resulting solution oscillatory, must be $-C_2$. Also, since $\phi(x)$ was assumed to have a simple zero at $x = 0$, the argument of $[-\phi(x)]^{-1/4}$ is increased by $\pi/4$ as arc L is traversed. This implies that for $x = +R$ the WKB solution corresponding to (56) has the form

$$\tilde{w}_r = 2\, C_2\, [\phi(x)]^{-1/4} \cos\left\{\lambda \int_0^x \sqrt{\phi(\zeta)}\,d\zeta - \pi/4\right\}. \tag{58}$$

Thus Zwaan obtained the connection formula:

$$[-\phi(x)]^{-1/4} \exp\left\{-\lambda \int_0^x \sqrt{\phi(\zeta)}\,d\zeta\right\} \leftrightarrow 2[\phi(x)]^{-1/4} \cos\left\{\lambda \int_0^x \sqrt{\phi(\zeta)}\,d\zeta - \pi/4\right\}. \tag{59}$$

Zwaan's approach, which avoided crossing the turning point, is similar to that of Stokes [1856; 1864]. But in contrast to Stokes who used the actual solutions together with the asymptotic solution, Zwaan used only the WKB solutions. He employed analytic continuation on the assumption that $\phi(x)$ can be approximated to some degree by an analytic function. But since the term "approximate" is not defined by Zwaan, it is difficult to ascertain if a certain function satisfied this requirement. In any event, Zwaan's assumptions placed strong restrictions on $\phi(x)$. Furthermore, it is well known that on an interval of the real line, a continuous function can be approximated to an arbitrary degree by an analytic function and this relationship will be maintained by their respective continuations to a small neighborhood of the interval. But in general that neighborhood becomes smaller as the degree of the approximation becomes larger, which would prevent Zwaan's condition (ii) from being satisfied. Zwaan's method lacks rigor. It was later modified and made rigorous in Fröman and Fröman [1965].

The connection relations described in this section were used by some authors to cross two turning points in succession. The procedure required that in the neighborhood of each turning point the various requirements be satisfied. If this can be done, the points are called "well separated." If the turning points are too close, then other procedures must be applied (see Heading [1962]).

The various approaches discussed in this paper to deal with the connection problem are innovative and daring. Though they provided useful answers, they are far from rigoruos. This problem was subsequently approached in a different and rigorous manner by R. E. Langer and his students during the 1930's. For a brief survey of Langer's method see McHugh [1971].

NOTES

1. For large values of the argument, Hankel's asymptotic expansion gives the one term asymptotic expansions (large σ) [Hankel 1868],

$$\sqrt{x}\, J_n(\sigma) \sim \sqrt{x}\,\sqrt{2/\pi\sigma}\, \cos\,[\sigma - (\pi/4) - (n\pi/2)]$$

$$\sqrt{x}\, I_n(\sigma) \sim [\sqrt{x}/\sqrt{2\pi\sigma}\,]\, \exp\{\sigma\} + \exp\,\{i(n + 1/2)\pi\}\,(1/\sqrt{2\pi\sigma})\, \exp\{-\sigma\}.$$

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THIRD INTERNATIONAL LEIBNIZ CONGRESS

Under the title "*Theoria cum praxi*, the relation of theory and practice in the XVII and XVIII centuries," the Congress will meet in Hannover, 12-16 November 1977 under the auspices of the Gottfried-Wilhelm-Leibniz-Gesellschaft. Central themes will be: (1) the theoretical and practical aims of the new philosophical systems. (2) political philosophy and legal and political practice. (3) the relation of theory and practice in the natural sciences and the role of scientific organizations. (4) the reconciliation of the Christian Churches as a theoretical and practical problem.

Since 1977 is the 300th anniversary of Spinoza's death and of the first publication of his *Ethica ordine geometrico demonstrata*, his thought and influence will receive special attention. During the Congress it will be possible to inspect the Leibniz manuscripts in the Niedersächsische Landesbibliothek.

Enquiries should be addressed to the Tagungsbüro, Niedersächsische Landesbibliothek, Waterloostrasse 8, D-3000 Hannover 1, FRG.